3) Quasiconformal surgery

Holomorphic maps are quite rigid: by analytic continuation, the map is determined in its whole domain of definition (if connected) by the values it takes on a small open subsets (or just a set with an accumulation point).

In particular, holomorphic change of coordinates are quite rigid too, think of $\mathrm{Aut}(\hat{\mathbb{C}}) \simeq \mathbb{C}(z)$, where an element is completely determined by the value it takes in three points.

On the other hand, homeomorphisms, or even C^∞ diffeomorphisms, are quite flexible, and in general, the conjugation by this kind of maps does not preserve the class of holomorphic maps.

Quasi-conformal maps provide the right setting: they are flexible enough to allow surgery, and forge new dynamical systems with properties ~~in~~ some way combining given dynamical systems, and rigid enough that we can control when, conjugating a holomorphic map with a quasiconformal automorphism, we ~~..~~ get another holomorphic map.

Let now f be a C^1 homeomorphism defined in a region in $\mathbb{C}$. Or more generally, in a open $U \subset X$ Riemann Surface

Recall that, setting $z = x+iy$, $\frac{\partial f}{\partial z} =: f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$; $\frac{\partial f}{\partial \bar{z}} =: f_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$

The differential df can be written in these coordinates as $df = f_z\, dz + f_{\bar{z}}\, d\bar{z}$

and the Jacobian is: $J_f = |f_z|^2 - |f_{\bar{z}}|^2$.

In particular, f preserves the orientation if and only if $|f_z| > |f_{\bar z}|$.

We will assume this is the case from now on.

Given f as above, we may define:

$\mu_f := \frac{f_{\bar z}}{f_z}$, called Beltrami coefficient (or complex dilatation of f)

Notice that $|\mu_f| < 1$ in our assumption.

We are also assuming ~~f~~ $f_z \neq 0$ $\forall p \in U$ $^{\text{a domain}}$; if not, we would have

$f_z(p) = f_{\bar z}(p) = 0$, and f is not locally invertible at p.

The associated equation $f_{\bar z} = \mu f_z$ is called the Beltrami equation

(associated to the coefficient μ).

Notice that $\mu_f \equiv 0 \iff f_{\bar z} \equiv 0 \iff f$ is holomorphic.

We know that a holomorphic injective map is exactly a conformal map,

i.e., it preserves infinitesimal angles (i.e., the differential df preserves angles

on the tangent space (at 0)).

Analogously, f preserves infinitesimal circles.

(a $U \subset X$ × Riemann Surface)

Def: Let $U \subset \mathbb{C}$ be a (open connected) domain, and $f : U \to \mathbb{C}$ a C^1 map and homeo.

f is called k-quasiconformal, with $k \in [0,1)$, constant, if

its Beltrami coefficient μ_f is bounded by k: $\|\mu_f\| \leq k$.

Denote the set of k-quasi-conformal maps $QC(k, U)$.

This definition can be generalized by only asking that derivatives exist as

distributions, μ is measurable, L^∞, and $\|\mu\|_\infty < 1$.

Quasiconformal maps in this setting need to be locally L^2. (and C^0 homeos)

As holomorphic maps send ~~circles~~ infinitesimal circles to circles, quasi-conformal maps send infinitesimal ellipses to circles (ellipses).

By direct computation, the preimage through df_p of circles of $T_{f(p)}\mathbb{C}$ give ellipses of ratio of length of axes depending only on μ, and given by $K = \dfrac{1+|\mu|}{1-|\mu|} \geq 1$

In fact, circles of $T_{f(p)}\mathbb{C}$ are given by $w\bar{w} = r > 0$, and through the pull-back, using $f_{\bar{z}} = \mu f_z$, we get $|f_z|^2 (z + \mu\bar{z})(\overline{z+\mu\bar{z}}) = r$

It ~~follows~~ when rewriting $z + \mu\bar{z} = z\underbrace{\left(1 + \mu\frac{\bar{z}}{z}\right)}_{*}$, we notice that $(*)$ varies between $1+|\mu|$ and $1-|\mu|$, from which we get the value of K.

Inverting this relation, we get $|\mu| = \dfrac{K-1}{K+1}$.

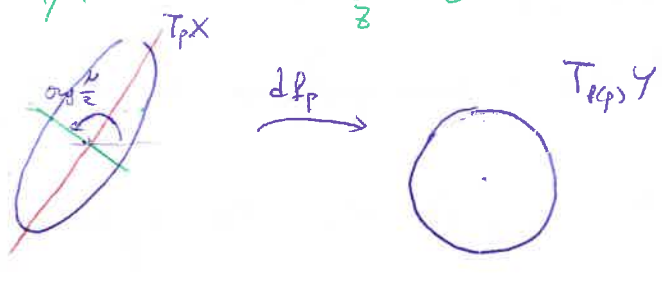
It is also possible to find the directions of the two principal axes, given by the arguments $\dfrac{\pi}{2} + \arg\left(\dfrac{\mu}{2}\right)$ and $\arg\left(\dfrac{\mu}{2}\right)$

$|z| = \dfrac{c}{1 \mp |\mu|}$

The equation is $|z + \mu\bar{z}| = \text{const}$, the ~~max~~ min modulus is obtained when $|z+\mu\bar{z}| = (1 \mp |\mu|)|z|$; max / min

$$1 + \mu\frac{\bar{z}}{z} = 1 + |\mu| \iff \frac{\mu}{|\mu|} = e^{i\arg\mu} = \frac{z}{\bar{z}} = e^{2i\arg z} \rightarrow \arg z = \arg\frac{\mu}{2}$$

$$1 + \mu\frac{\bar{z}}{z} = 1 - |\mu| \iff -\frac{\mu}{|\mu|} = e^{i(\arg\mu + \pi)} = \frac{z}{\bar{z}} = e^{2i\arg z} \Rightarrow \arg z = \frac{\pi}{2} + \arg\frac{\mu}{2}$$



This pictures comes for any point $p \in U$: given a Beltrami coefficient μ (μ measurable, $\mu \in L^\infty(U)$, $\|\mu\|_\infty < 1$), we may associate to almost every $p \in U$ a "pencil" of ellipses in T_pU, given by the preimages through df_p of circles in $T_{f(p)}\mathbb{C}$, which for what we saw are the ellipses of ratio of semiaxes given by $\frac{1+|\mu|}{1-|\mu|}$ and semiaxes directions $\frac{\pi}{2} + \arg\frac{\mu}{2}$ and $\arg\frac{\mu}{2}$.

This datum is called a "field of ellipses".

To sum up:

- A k-quasiconformal map $f : X \to Y$ (X, Y Riemann surfaces) is a homeomorphism, almost everywhere C^1, (with distributional derivatives locally in L^2) which admits a
- measurable Beltrami coefficient $\mu = \frac{f_{\bar z}}{f_z}$ (defined almost everywhere), $\mu \in L^\infty(X)$, $\|\mu\|_\infty \leq k < 1$.
- To this data is associated a field of ellipses on TX.

We study now the behavior of quasi-conformal mappings under composition.

Prop: Let f, g be quasiconformal maps. Then,

(1) $$\mu_{g \circ f} = \frac{\mu_f \cdot A_f + \mu_g \circ f}{A_f + \mu_g \circ f \cdot \overline{\mu_f}}, \quad A_f = \left(\frac{f_z}{|f_z|}\right)^2 \underset{= \frac{f_z}{\bar{f}_{\bar z}}}{} \quad (|A_f| = 1)$$

In particular, if f is k-quasiconformal and h is holomorphic (conformal), we get:

(2) $\mu_{h \circ f}(z) = \mu_f(z)$, (3) $\mu_{g \circ h}(z) = \mu_g \circ h(z) \cdot \frac{\overline{h'(z)}}{h'(z)}$

and $\Lambda = \frac{1}{A_f} \in \partial\mathbb{D}$. In particular, the composition of quasiconformal maps is itself quasiconformal.

Corollary. If f and g are quasiconformal with the same Beltrami coefficient, then $g = \Phi \circ f$, with Φ conformal

Proof: Consider $\Phi = g \circ f^{-1}$. Since f is quasiconformal of Beltrami coefficient $\mu = \mu_f$, df_p sends the field of ellipses defined by $(\mu)_p$ to the field of circles

Then f^{-1} sends infinitesimal circles to the field of ellipses determined by μ, which is sent by g to infinitesimal circle. Hence Φ is holomorphic. $\square$

Can be also done using the previous proposition: $\mu_{f^{-1}} \circ f = -\mu_f \cdot A_f$.

$$\mu_{g\circ f^{-1}} = \frac{A_f \cdot \mu_f + \mu_{g^{-1}} \circ f}{A_f + \overline{\mu_f} \cdot \mu_{g^{-1}} \circ f} \underset{\mu_g = \mu_f = \mu}{=} \frac{A_f \mu - \mu \cdot A_f}{A_f (1 - \mu\bar{\mu})} = 0.$$

We will focus on quasi-conformal maps on $\hat{\mathbb{C}}$. In this case, if μ is the Beltrami coefficient of two quasiconformal maps f, g on $\hat{\mathbb{C}}$, then $g \circ f^{-1} \in \mathrm{Aut}(\hat{\mathbb{C}})$

In particular, the value of a quasiconformal map with given Beltrami coefficient is uniquely determined by fixing the values of $0, 1, \infty$ {(for example $f(0)=0$, $f(1)=1$, $f(\infty)=\infty$). We denote this choice with Φ_μ, and refer to it as a "normalised" solution of the Beltrami equation.

The formulas expressed above suggest a different approach to Beltrami coefficients. We can in fact consider Beltrami forms i.e., $(-1,1)$-forms written locally as $\mu(z)\frac{d\bar{z}}{dz}$, with $\mu(z)$ L^∞ and $\|\mu\|_\infty < 1$.

If $z = h(w)$, h holomorphic then the pull back of this forms is:

Finally: (4) $\mu_{f^{-1}}(f(z)) = -\mu_f(z)\cdot A_f(z)$.

Proof: by the chain rule, we have,

$$\mu_{g\circ f} = \frac{\frac{\partial g\circ f}{\partial \bar z}}{\frac{\partial g\circ f}{\partial z}} = \frac{\frac{\partial g}{\partial z}\circ f\cdot\frac{\partial f}{\partial \bar z} + \frac{\partial g}{\partial \bar z}\circ f\cdot\frac{\partial \bar f}{\partial \bar z}}{\frac{\partial g}{\partial z}\circ f\cdot\frac{\partial f}{\partial z}+\frac{\partial g}{\partial \bar z}\circ f\cdot\frac{\partial \bar f}{\partial z}} = \frac{\frac{\partial f}{\partial \bar z}+\mu_g\circ f\,\frac{\partial \bar f}{\partial \bar z}}{\frac{\partial f}{\partial z}+\mu_g\circ f\cdot\frac{\partial \bar f}{\partial z}} = (*)$$

In particular, if $\mu_g=0$, we get. $\mu_{g\circ f}(z) = \mu_f(z)$. (2)

In fact h sends infinitesimal circles to circles, so $\mu_{h\circ f}=\mu_f$.

Notice that $\frac{\partial f}{\partial z} = \overline{\left(\frac{\partial \bar f}{\partial \bar z}\right)}$. By dividing numerator and denominator by $\frac{\partial f}{\partial z}$ we get

$$\mu_{g\circ f} = \frac{\frac{\partial f}{\partial \bar z}\Big/\frac{\partial \bar f}{\partial \bar z}+\mu_g\circ f}{\frac{\partial f}{\partial z}\Big/\frac{\partial \bar f}{\partial \bar z}+\mu_g\circ f\,\frac{\partial \bar f}{\partial z}\Big/\frac{\partial \bar f}{\partial \bar z}} = \frac{\mu_f\cdot\frac{\partial f}{\partial z}\Big/\frac{\partial \bar f}{\partial \bar z}+\mu_g\circ f}{\frac{\partial f}{\partial z}\Big/\frac{\partial \bar f}{\partial \bar z}+\mu_g\cdot\overline{\frac{\partial f}{\partial \bar z}}\Big/\frac{\partial f}{\partial z}} = \frac{\mu_f\cdot A_f+\mu_g\circ f}{A_f+\mu_g\circ f\cdot\overline{\mu_f}},$$

with $A_f = \frac{\partial f}{\partial z}\Big/\frac{\partial \bar f}{\partial \bar z} = \frac{f_z}{\overline{f_z}} = \left(\frac{f_z}{|f_z|}\right)^2$.

This gives (1). By applying this when $\mu_f=0$, we get:

$$\mu_{g\circ f} = \frac{\mu_g\circ f}{A_f} = \qquad \text{with } \frac{1}{A_f} = \frac{\partial \bar f}{\partial \bar z}\Big/\frac{\partial f}{\partial z} = \frac{\overline{f'(z)}}{f'(z)}. \quad (3).$$

Finally, applying (1) for $g=f^{-1}$, we get $0 = A_f\cdot\mu_f+\mu_{f^{-1}}\circ f$, which gives (4). □

Rem: Notice in particular that if f K-quasiconformal and h is holomorphic (conformal), then f^{-1}, $h\circ f$ and $f\circ h$ are K-quasiconformal. In fact, the expression for $\mu_{g\circ f}$ can be rewritten as $\mu_{g\circ f} = \Lambda\cdot\frac{Z+\alpha}{1+\bar\alpha Z}$, with $Z=\mu_g\circ f$, $\alpha=A_f\cdot\mu_f\in\mathbb{D}$

$$\mu(h(w))\,\frac{\overline{\partial h(w)}}{\partial h(w)} = \mu(h(w))\,\frac{\overline{h'(w)}}{h'(w)}\,\frac{d\bar w}{dw} = \mu_{f\circ h}(w)\,\frac{d\bar w}{dw}.$$

In particular, the pull-back of a Beltrami ~~form~~ coefficient correspond to the pullback of a Beltrami form, seen as a $(-1,1)$-form.

Push forward and pull back.

The formulas above allow to define the pull-back of a Beltrami coefficient (or form, as explained above) through a holomorphic map. In fact, if μ is a Beltrami coefficient, ~~and~~ defined on some open domain U, and $f: V \to U$ is a holomorphic map, then the pull back of μ is defined by $(f^*\mu)(z) = \mu(f(z))\,\dfrac{\overline{f'(z)}}{f'(z)}$.

This is well defined outside $C(f)$, which is a discrete set, hence of measure zero. These formulas can be also used to define the pull back of a Beltrami form by a quasiconformal map.

When $f: U \to W$ is ~~an~~ a biholomorphism, we can define the pushforward of μ as $f_*\mu = (f^{-1})^*\mu$.

The measurable Riemann mapping theorem. (Ahlfors-Bers).

Theorem: let μ be a Beltrami coefficient (or form) on $\hat{\mathbb{C}}$.

Then there exists a unique quasiconformal homeomorphism $\Phi_\mu : \hat{\mathbb{C}} \circlearrowleft$ normalized (by $\Phi_\mu\begin{pmatrix}0\\1\\\infty\end{pmatrix} = \begin{matrix}0\\1\\1\end{matrix}$), satisfying the Beltrami equation

$$\frac{\partial \Phi_\mu}{\partial \bar z} = \mu \cdot \frac{\partial \Phi_\mu}{\partial z}$$ (almost everywhere, locally in L^2 in the distributional sense)

Fact: if μ, $\frac{\partial\mu}{\partial z}$ are C^1, then so is Φ_μ.

Interpretation in terms of μ-conformal structures.

Let X be a Riemann surface, and μ be a Beltrami coefficient.

μ induces a structure of Riemann surface on X, denoted $X[\mu]$, that can be seen as a twisted Riemann surface structure. Its atlas $\mathcal{A}(\mu)$ is given by all the maps $\varphi: U \to \hat{\mathbb{C}}$ which are quasiconformal with Beltrami coefficient μ. Since two quasiconformal maps with the same Beltrami coefficient differ by a biholomorphism, and local solutions exist by the measurable Riemann mapping theorem, $\mathcal{A}(\mu)$ defines a Riemann surface structure.

Def: a map $f: X \to \hat{\mathbb{C}}$ is μ-analytic if $f: X[\mu] \to \hat{\mathbb{C}}$ is analytic. If f is also an homeomorphism, we say that f is μ-conformal.

Rem: TFAE: $\varphi: X \to \hat{\mathbb{C}}$.

1) φ is μ-conformal ; 2) $\varphi \in \mathcal{A}(\mu)$ 3) $\varphi: X[\mu] \to \hat{\mathbb{C}}$ is analytic.

More generally, we can consider X, Y domains $(\subset \hat{\mathbb{C}})$, and $f: X \to Y$ analytic. Consider μ Beltrami coefficient in X, ν in Y.

Then: $f: X[\mu] \to Y[\nu]$ is analytic

$$\Updownarrow$$

$$\nu(f(z)) = \frac{f'(z)}{\overline{f'(z)}} \cdot \mu(z) \quad \forall z \in X \iff f^*\nu = \mu. \text{ (almost everywhere)}$$

The proof is only a matter of unravelling definitions.

Rational maps and quasiconformal conjugacy.

We now focus on rational maps $f : \hat{\mathbb{C}} \circlearrowleft$. We ask when, by conjugating by a quasiconformal homeomorphism $\Phi : \hat{\mathbb{C}} \circlearrowleft$, we end up with a holomorphic (hence rational) map $\tilde{f} = \Phi \circ f \circ \Phi^{-1} : \hat{\mathbb{C}} \circlearrowleft$.

Proposition. $f : \hat{\mathbb{C}} \circlearrowleft$ rational, $\Phi : \hat{\mathbb{C}} \circlearrowleft$ quasiconformal of Beltrami coefficient μ. Then $\tilde{f} = \Phi \circ f \circ \Phi^{-1}$ is rational if and only if:

$$\mu(f(z)) = \frac{\overline{f'(z)}}{f'(z)} \mu(z) \qquad (\text{i.e., } f^* \mu = \mu) \quad \text{almost everywhere on } \hat{\mathbb{C}}.$$

In this case, $\deg \tilde{f} = \deg f$.

Proof: The condition $f^* \mu = \mu$ is equivalent to $\hat{\mathbb{C}}[\mu] \xrightarrow{f} \hat{\mathbb{C}}[\mu]$ being holomorphic.

Being Φ a chart of $\hat{\mathbb{C}}[\mu]$, this is equivalent of having $\Phi \circ f \circ \Phi^{-1}$ a holomorphic map on $\hat{\mathbb{C}}$.

The relation on the degree comes from the fact that, being $\deg(f)$ countable as the number of preimages of a generic point, is a topological invariant of conjugacy.

This proposition allows to deform rational maps through quasiconformal conjugation.

Quasiconformal surgery

A useful application of quasiconformal deformation is quasiconformal surgery

Def: a homeomorphism $g: \hat{\mathbb{C}} \circlearrowleft$ is called quasiregular if it is the composition $g = f \circ \Psi$ of f holomorphic and Ψ a quasiconformal homeomorphism.

Theorem: Let $g: \hat{\mathbb{C}} \circlearrowleft$ be quasiregular, and $E \subset \hat{\mathbb{C}}$ an open such that

- $g(E) \subseteq E$
- g is analytic on E and on $\hat{\mathbb{C}} \setminus g^{-1}(E)$.

Then there exists $\Phi: \hat{\mathbb{C}} \circlearrowleft$ quasiconformal so that $\Phi \circ g \circ \Phi^{-1}$ is rational.

Proof: Since $g(E) \subseteq E$, we have $E \subseteq g^{-1}(E) \subseteq g^{-2}(E) \subseteq \dots$

Set $E_0 = E$, $E_1 = g^{-1}(E) \setminus E$, $E_{k+1} = g^{-(k+1)}(E) \setminus g^{-k}(E)$, ~~and~~ ~~...~~ so that $GO(E) = \bigsqcup_{k \geq 0} E_k$. We also set $K = \hat{\mathbb{C}} \setminus GO(E)$.

We define a Beltrami coefficient μ on $\hat{\mathbb{C}}$ by setting $\mu \equiv 0$ on E_0, $\mu = (g^n)^* 0$ (as Beltrami forms) on E_n, and again $\mu \equiv 0$ on K because $g|_{E_0}$ is holomorphic

In formula: $\mu = \mu_g$ on E_1, and recursively $\mu(z) = \mu(g(z)) \left(\frac{\overline{g_z(z)}}{g_z(z)} \right)$ $\forall z \in E_n$, $n \geq 2$.

By Ahlfors Bers theorem we get a Φ normalised solution of the Beltrami equation associated to μ.

To show that $\Phi \circ g \circ \Phi^{-1}$ is analytical we must show that $g^* \mu = \mu$ as Beltrami forms. But this holds by construction. $\square$